

# Note About Classical Dynamics of Pure Spinor String on $AdS_5 \times S_5$ Background

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**ABSTRACT:** We will discuss some properties of the pure spinor string on the  $AdS_5 \times S_5$  background. Using the classical Hamiltonian analysis we will show that the vertex operator for the massless state that is in the cohomology of the BRST charges describes on-shell fluctuations around  $AdS_5 \times S_5$  background.

**KEYWORDS:** string theory.

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## 1. Introduction and summary

The string action that posses manifest space time supersymmetry, is famous Green-Swarz (GS) action [1]. However, no one has succeeded in making a covariant quantisation of GS action. The source of the difficulty is impossibility to achieve the separation of the fermionic first class and second class constraints associated with local  $\kappa$  symmetry in a manifestly covariant way. As in ten dimensions the smallest covariant spinor corresponding to Majorana-Weyl spinor has 16 real components, 8 first class and 8 second class constraints (that arise in heterotic or Type I GS superstrings) do not fit into such covariant spinor representation separately.

It was soon realised that second class constraints are present in the GS action due to the definition of the conjugate momenta to  $\theta$ . By using proposal of Faddeev and Frandkin one could turn second class constraints into the first class ones by adding further fields but upon quantisation one now obtained infinite set of ghost-for-ghosts that in the end is infinite [2, 3].

Few years ago Berkovits has proposed an interesting approach to covariant quantisation of superstrings, using pure spinors [4, 5, 6, 7]. The Berkovits approach seems to be the right directions for covariant quantisation of the GS action as was shown on many examples in the past. In fact, it appears that the statement that the pure spinor approach provides a super-Poincare covariant quantization of superstring theories is correct but deserves a warning. Since this approach is based on very simple form of the BRST operator is is not obvious how it can be obtained by gauge fixing a reparametrisation invariant worldsheet action. On the other hand there exist some proposals to find more geometrical version of the pure spinor formalism

[14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 8] and for recent discussion of these approaches we recommend the paper [31] where new promising formulation of the pure spinor formalism was also proposed. In summary, the super-Poincare covariant formalism can be applied in many situations, for example in the computation of the multi loop scattering amplitudes [32, 33, 34, 35] and the formulation of the quantum superstring in the  $AdS_5 \times S_5$  Ramond-Ramond background.

The superstring worldsheet action in  $AdS_5 \times S_5$  Ramond-Ramond background can be studied at the classical level using either the GS formalism [9] or the pure spinor formalism [10]. Since  $AdS_5 \times S_5$  is solution of type IIB supergravity the worldsheet action is classically  $\kappa$ -invariant in the GS formalism and is classically BRST invariant in the pure spinor formalism [12]. Moreover, it was even shown in [13] that the BRST invariance of the pure spinor action persists at the quantum level.

However even if it was shown that the pure spinor action in  $AdS_5 \times S_5$  with Ramond-Ramond flux leads to well defined CFT this theory has not been solved yet. The main difficulty is in the fact that we cannot separate worldsheet fields to their holomorphic and antiholomorphic parts. The peculiar property of the Ramond-Ramond background can be nicely seen in the analysis of string theory on  $AdS_3 \times S_3$  background. While the case with nonzero  $NS$  two form field is well known understood the the analysis of the  $AdS_3 \times S_3$  with Ramond-Ramond flux is much more involved [30].

Due to these facts we mean that it is useful to collect as far as informations about the pure spinor string theory in  $AdS_5 \times S_5$  background as possible. In particular we mean that the classical canonical analysis of the covariant string in  $AdS_5 \times S_5$  could be helpful for the study of the quantum conformal field theory. This paper is the first step in the study of the properties of the classical covariant string in the  $AdS_5 \times S_5$  background.

The rest of this paper is organised as follows. In the next section (2) we review the pure spinor approach in the flat background with emphasis on the worldsheet covariant formulation that allows classical Hamiltonian analysis.

In section (3) we will study the properties of the pure spinor action in general background that was introduced in [12]. We again formulate this action in manifest covariant formalism even if we have to stress that we work on the flat worldsheet. We express the BRST charge in terms of the canonical variables of the theory and then we determine the action of this charge on the function on the extended phase space and that can be interpreted as the vertex operator for massless state of the theory. Then we show that the requirement that this vertex is in the cohomology of the BRST charges leads to the result that this function describes massless fluctuations around given background that obey appropriate linearised equations of motion.

In section (4) we begin to discuss the classical pure spinor action on  $AdS_5 \times S_5$ . We review the constructions of BRST charges performed in [10, 11] with emphasis

on the manifest worldsheet covariant formulation. We also determine the equations of motions for matter and ghost fields. Then we express the BRST charge in terms of canonical variables and we will calculate its action on function defined on the phase space and that is classical analogue of the vertex operator in the standard treatment. We again argue that the requirement that this function belongs to the cohomology of BRST charges implies that these vertex operators correspond to on-shell fluctuations on the  $AdS_5 \times S_5$  background.

The summary of this paper is as follows. We explicitly determine the action of the BRST charge in general background on function on the phase space and we will argue that in general the requirement that this function is in the cohomology of the BRST charges implies that this function describes on-shell fluctuations around given supergravity background. We obtain the result that is in the agreement with the action of the BRST charge on these vertex operators that was suggested in [10].

We would like also stress that this paper gives only modest contribution to the study of the classical canonical formalism of pure spinor action in  $AdS_5 \times S_5$ . As a next step we would like to determine the algebra of the currents that defines given action. Then the knowledge of this algebra will be helpful for construction of more general functions on the phase spaces that could serve as an analogues of the vertex operators for massive states on the  $AdS_5 \times S_5$  background. This work is currently progress.

## 2. Pure Spinor Superstring in Flat Background

In this section we give a brief review of the pure spinor formalism in flat spacetime. For more details, see [24, 25, 26, 27, 28].

In a flat background, the worldsheet action in the pure spinor formalism takes the form <sup>2</sup>

$$S = - \int d^2x \sqrt{-\eta} \left( \frac{1}{2} \eta^{\mu\nu} \partial_\mu Y^m \partial_\nu Y^n \eta_{mn} + p_{\mu\alpha} P^{\mu\nu} \partial_\nu \theta^\alpha + \hat{p}_{\mu\hat{\alpha}} \tilde{P}^{\mu\nu} \partial_\nu \hat{\theta}^{\hat{\alpha}} + \right. \\ \left. + w_{\mu\alpha} P^{\mu\nu} \partial_\nu \lambda^\alpha + \hat{w}_{\mu\alpha} \tilde{P}^{\mu\nu} \partial_\nu \hat{\lambda}^{\hat{\alpha}} \right), \quad (2.1)$$

where

$$\eta_{\mu\nu} = \text{diag}(-1, 1), \quad \sqrt{-\eta} = \sqrt{-\det \eta}, \\ P^{\mu\nu} = \eta^{\mu\nu} - \epsilon^{\mu\nu}, \quad \tilde{P}^{\mu\nu} = \eta^{\mu\nu} + \epsilon^{\mu\nu}, \quad P^{\mu\nu} = \tilde{P}^{\nu\mu}, \\ \epsilon^{\mu\nu} = \frac{e^{\mu\nu}}{\sqrt{-\eta}}, \quad e^{01} = -e^{10} = 1. \quad (2.2)$$

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<sup>2</sup>We work in units  $2\pi\alpha' = 1$ .

We label the worldsheet with variables  $x^\mu$ ,  $x^0 = t$ ,  $x^1 \equiv x$ . The flat spacetime indices are labeled with  $m, n = 0, \dots, 9$ , spinor indices are labeled with  $\alpha, \hat{\alpha} = 1, \dots, 16$ . Finally  $\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}}$  are bosonic ghosts satisfying the pure spinor conditions

$$\lambda^\alpha(\gamma^m)_{\alpha\beta}\lambda^\beta = 0, \hat{\lambda}^{\hat{\alpha}}(\gamma^m)_{\hat{\alpha}\hat{\beta}}\hat{\lambda}^{\hat{\beta}} = 0, \text{ for } m = 0 \text{ to } 9, \quad (2.3)$$

where  $\gamma_{\alpha\beta}^m, \gamma_{\hat{\alpha}\hat{\beta}}^m$  are  $16 \times 16$  symmetric matrices that are off-diagonal components of the  $32 \times 32$  gamma matrices. Finally, as we will see below,  $w_{\mu\alpha}, \hat{w}_{\mu\hat{\alpha}}$  are related to the momentum conjugate to  $\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}}$ . At the same time  $p_{\mu\alpha}, \hat{p}_{\mu\hat{\alpha}}$  are related to the momenta conjugate to  $\theta^\alpha, \hat{\theta}^{\hat{\alpha}}$ .

Fundamental object of the pure spinor formalism are BRST charges  $Q_R, Q_L$

$$Q_L = \int dx j_L^0, Q_R = \int dx j_R^0, \quad (2.4)$$

where

$$\begin{aligned} j_L^\mu &= \lambda^\alpha \tilde{P}^{\mu\nu} d_{\nu\alpha} = \lambda^\alpha d_{\nu\alpha} P^{\nu\mu}, \\ j_R^\mu &= \hat{\lambda}^{\hat{\alpha}} P^{\mu\nu} \hat{d}_{\nu\hat{\alpha}} = \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\nu\hat{\alpha}} \tilde{P}^{\nu\mu}. \end{aligned} \quad (2.5)$$

Since  $Q_{R,L}$  have to be time independent the currents (2.5) are conserved

$$\partial_\mu j_{R,L}^\mu = 0. \quad (2.6)$$

Even if the notation used above is slightly unconventional it is equivalent to the standard one. Namely, performing the Wick rotation on the worldsheet and then introducing chiral variables  $z, \bar{z}$  it is easy to see that the only nonzero components of the projectors  $P, \tilde{P}$  are  $P^{z\bar{z}} = 1, \tilde{P}^{\bar{z}z} = 1$ . Then the equation (2.6) is equal to

$$\partial_{\bar{z}}(\lambda^\alpha d_{\alpha z}) = 0, \partial_z(\hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha} \bar{z}}) = 0 \quad (2.7)$$

that allow us to construct two conserved charges

$$Q_L = \frac{1}{2\pi i} \oint dz \lambda^\alpha d_{\alpha z}, Q_R = \frac{1}{2\pi i} \oint d\bar{z} \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha} \bar{z}} \quad (2.8)$$

that are the standard forms of the BRST charges used in the literature. Since in what follows we will study the pure spinor action using Hamiltonian formalism we use the convention given in (2.4). We must also stress that in the flat spacetime the objects  $d_{\mu\alpha}, \hat{d}_{\mu\hat{\alpha}}$  take the form

$$\begin{aligned} d_{\mu\alpha} &= p_{\mu\alpha} + (i\partial_\mu Y^m + \frac{1}{2}\theta^\alpha(\gamma^m)_{\alpha\beta}\partial_\nu\theta^\beta + \frac{1}{2}\hat{\theta}^{\hat{\alpha}}(\gamma^m)_{\hat{\alpha}\hat{\beta}}\partial_\nu\hat{\theta}^{\hat{\beta}})(\gamma_m\theta)_\alpha, \\ \hat{d}_{\mu\hat{\alpha}} &= \hat{p}_{\mu\hat{\alpha}} + (i\partial_\mu Y^m + \frac{1}{2}\theta^\alpha(\gamma^m)_{\alpha\beta}\partial_\nu\theta^\beta + \frac{1}{2}\hat{\theta}^{\hat{\alpha}}(\gamma^m)_{\hat{\alpha}\hat{\beta}}\partial_\nu\hat{\theta}^{\hat{\beta}})(\gamma_m\hat{\theta})_{\hat{\alpha}}. \end{aligned} \quad (2.9)$$

The physical states in the pure spinor formalism are defined as vertex operators of ghost number  $(1, 1)$  in the cohomology of the nilpotent BRST charges. The massless states are constructed from zero modes only and hence are represented by the vertex operator

$$U = \lambda^\alpha \hat{\lambda}^{\hat{\beta}} A_{\alpha\hat{\beta}}(Y^m, \theta, \hat{\theta}) , \quad (2.10)$$

where  $A_{\alpha\hat{\beta}}$  is bispinor  $N = 2$   $D = 10$  superfield which depends on the worldsheet zero modes of  $y^m$ ,  $\theta^\alpha$  and  $\hat{\theta}^{\hat{\alpha}}$  only.

In terms of the BRST analysis of the classical theory this condition translates to the requirement that the graded Poisson bracket <sup>3</sup> between  $Q_{L,R}$  and  $U$  is equal to zero

$$\{Q_L, U\} = \{Q_R, U\} = 0 . \quad (2.11)$$

It is clear that shift of the function  $U$  in the form

$$\delta U = \{Q_L, \Lambda\} + \{Q_R, \hat{\Lambda}\} , \quad (2.12)$$

where  $\{Q_R, \Lambda\} = \{Q_L, \hat{\Lambda}\} = 0$  does not change the equations (2.11). Applying these conditions to  $U$  given in (2.10) and if we also define  $\Lambda, \hat{\Lambda}$  as

$$\Lambda = \hat{\lambda}^{\hat{\alpha}} \hat{\Gamma}_{\hat{\alpha}} , \hat{\Lambda} = \lambda^\alpha \Gamma_\alpha \quad (2.13)$$

and using the fact that pure spinors satisfy

$$\lambda^\alpha \lambda^\beta = \frac{1}{3840} (\lambda \gamma^{mnpqr} \lambda) \gamma_{mnpqr}^{\alpha\beta} , \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} = \frac{1}{3840} (\hat{\lambda} \gamma^{mnpqr} \hat{\lambda}) \gamma_{mnpqr}^{\hat{\alpha}\hat{\beta}} \quad (2.14)$$

we find that  $A_{\alpha\hat{\beta}}$  satisfy the conditions

$$\gamma_{mnpqr}^{\alpha\beta} D_\alpha A_{\beta\hat{\gamma}} = \gamma_{mnpqr}^{\hat{\alpha}\hat{\beta}} D_{\hat{\alpha}} A_{\gamma\hat{\beta}} = 0 \quad (2.15)$$

that together with gauge transformations

$$\delta A_{\beta\hat{\gamma}} = D_\beta \hat{\Gamma}_{\hat{\gamma}} + D_{\hat{\gamma}} \Gamma_\beta , \gamma_{mnpqr}^{\alpha\beta} D_\alpha \Gamma_\beta = \gamma_{mnpqr}^{\hat{\alpha}\hat{\beta}} D_{\hat{\alpha}} \hat{\Gamma}_{\hat{\gamma}} = 0 , \quad (2.16)$$

where  $D_\alpha = \frac{\partial}{\partial \theta^\alpha} + (\gamma^m \theta)_\alpha \partial_m$  ,  $D_{\hat{\alpha}} = \frac{\partial}{\partial \hat{\theta}^{\hat{\alpha}}} + (\gamma^m \hat{\theta})_{\hat{\alpha}} \partial_m$  are the supersymmetric derivatives of flat  $N = 2$   $D=10$  superspace. Performing now the analysis as in [10] one can argue that (2.15) and (2.16) correctly reproduce the Type IIB supergravity spectrum in flat spacetime.

After this brief review of the main properties of the pure spinor formalism in flat spacetime we focus in the next section on the formulation of the pure spinor string in general background.

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<sup>3</sup>Definition of the graded Poisson bracket will be given in next section.

### 3. Pure spinor action in General Background

The general form of the pure spinor action in Type IIA, IIB theories was given in [12, 24]. In what follows we concentrate on Type IIB theory and we write this pure spinor action in the manifest worldsheet covariant notation using  $\eta^{\mu\nu}$ ,  $\epsilon^{\mu\nu}$  and projectors  $P^{\mu\nu}$ ,  $\tilde{P}^{\mu\nu}$ . Then the pure spinor action in general background takes the form <sup>4</sup>

$$\begin{aligned}
S = & - \int d^2x \sqrt{-\eta} \left[ \frac{1}{2} \eta^{\mu\nu} g_{\mu\nu} - \frac{\epsilon^{\mu\nu}}{2} b_{\mu\nu} + \right. \\
& + P^{\alpha\hat{\beta}} d_{\alpha\mu} P^{\mu\nu} \hat{d}_{\hat{\beta}\nu} + d_{\alpha\mu} P^{\mu\nu} \partial_\nu Z^M E_M^\alpha(Z) + \hat{d}_{\hat{\alpha}\mu} \tilde{P}^{\mu\nu} \partial_\nu Z^M E_M^{\hat{\alpha}}(Z) + \\
& + \lambda^\alpha w_{\beta\mu} P^{\mu\nu} \partial_\nu Z^M \Omega_{M\alpha}^\beta + \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}\mu} \tilde{P}^{\mu\nu} \partial_\nu Z^M \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}} + \\
& \left. + C_\alpha^{\beta\hat{\gamma}} \lambda^\alpha w_{\beta\mu} \tilde{P}^{\mu\nu} \hat{d}_{\hat{\gamma}\nu} + \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma} \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}} P^{\mu\nu} d_{\gamma\nu} + S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}} \lambda^\alpha w_{\beta\mu} \hat{\lambda}^{\hat{\gamma}} P^{\mu\nu} \hat{w}_{\hat{\delta}\nu} \right] + S_\lambda + S_{\hat{\lambda}} , \\
S_\lambda = & - \int d^2x \sqrt{-\eta} w_{\mu\alpha} P^{\mu\nu} \partial_\nu \lambda^\alpha , S_{\hat{\lambda}} = - \int d^2x \sqrt{-\eta} \hat{w}_{\mu\hat{\alpha}} \tilde{P}^{\mu\nu} \partial_\nu \hat{\lambda}^{\hat{\alpha}} ,
\end{aligned} \tag{3.1}$$

where we have defined

$$g_{\mu\nu} = \partial_\mu Z^M E_M^a \partial_\nu Z^N E_N^b \eta_{ab} , b_{\mu\nu} = \partial_\mu Z^M \partial_\nu Z^N B_{MN} , \tag{3.2}$$

and where  $M = (m, \mu, \hat{\mu})$  are curved superspace indices,  $Z^M = (x^m, \theta^\mu, \theta^{\hat{\mu}})$ ,  $A = (a, \alpha, \hat{\alpha})$  are tangent superspace indices,  $S_\lambda, S_{\hat{\lambda}}$  are the flat actions for the pure spinor variables. Finally

$$E_M^\alpha, E_M^{\hat{\alpha}}, \Omega_{M\alpha}^\beta, \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}}, P^{\alpha\hat{\beta}}, C_\alpha^{\beta\hat{\gamma}}, \hat{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}, S_{\alpha\hat{\gamma}}^{\beta\hat{\delta}} \tag{3.3}$$

are background spacetime fields. Note also that  $d_{\mu\alpha}, \hat{d}_{\mu\hat{\alpha}}$  should be treated as an independent variables since  $p_{\mu\alpha}, \hat{p}_{\mu\hat{\alpha}}$  do not appear explicitly.

As in the flat space the fundamental object of the pure spinor formalism in the general background are the BRST operators  $Q_L, Q_R$  that again take the form

$$Q_L = \int dx \lambda^\alpha d_{\mu\alpha} P^{\mu 0} , Q_R = \int dx \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\mu\hat{\alpha}} \tilde{P}^{\mu 0} . \tag{3.4}$$

Our goal is to show that even at the classical level the requirement that the function that depends on  $Z^M$  only is in the cohomology of the BRST charges (3.4) implies that this function describes on-shell massless fluctuations around given supergravity background. Our analysis can be considered as modest contribution to the more general study performed in [12].

To use the classical formalism we have to express  $d_{\mu\alpha}, \hat{d}_{\mu\hat{\alpha}}$  in terms of canonical variables of the extended phase space with coordinates  $(Z^M, P_M, \lambda^\alpha, \pi_\alpha, \hat{\lambda}^{\hat{\alpha}}, \hat{\pi}_{\hat{\alpha}})$ ,

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<sup>4</sup>We omit the Fradkin-Tseytlin term  $\int \Phi(Z) r$  where  $\Phi$  is dilaton superfield and  $r$  is worldsheet curvature.

where  $P_M$  is a momentum conjugate to  $Z^M$  and  $\pi_\alpha, \hat{\pi}_{\hat{\alpha}}$  are momenta conjugate to  $\lambda^\alpha$  and  $\hat{\lambda}^{\hat{\alpha}}$  respectively. Then using the graded Poisson brackets we will calculate the action of this BRST operators (3.4) on the function on the extended phase space.

To proceed we firstly determine the momentum conjugate to  $Z^M$ . The conjugate momentum  $P_M$  is defined as the left variation of the action with respect to  $\partial_0 Z^M$

$$\begin{aligned} P_M = \frac{\delta^L S}{\delta \partial_0 Z^M} = & E_M^a \eta_{ab} E_N^b \partial_0 Z^N + \partial_1 Z^N B_{MN} - \\ & + E_M^\alpha d_{\alpha\mu} P^{\mu 0} + E_M^{\hat{\alpha}} d_{\hat{\alpha}\mu} \tilde{P}^{\mu 0} - \\ & - \lambda^\alpha w_{\alpha\mu} P^{\mu 0} \Omega_{M\alpha}^\beta - \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}\mu} \tilde{P}^{\mu 0} \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}} . \end{aligned} \quad (3.5)$$

Then we define canonical graded Poisson bracket between  $P_M$  and  $Z^M$  in the form

$$\{Z^M(x), P_N(y)\} = (-1)^{|M|} \delta_N^M \delta(x-y) , \quad (3.6)$$

where  $|X|$  is equal to one if  $X$  is Grassmann odd and  $|X| = 0$  if  $X$  is Grassman even. The graded Poisson bracket is defined as follows. Consider physical system with canonical bosonic variables  $q^i$  with corresponding conjugate momenta  $p_i$  together with fermionic variables  $\omega^\alpha$  with conjugate momenta  $\pi_\alpha$ <sup>5</sup>. Then the graded Poisson brackets involving fermionic systems is defined as

$$\{F, G\} = \left[ \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial \pi_i} - \frac{\partial F}{\partial \pi_i} \frac{\partial G}{\partial q^i} \right] + (-1)^{|F|} \left[ \frac{\partial^L F}{\partial \omega^\alpha} \frac{\partial^L G}{\partial \pi_\alpha} + \frac{\partial^L F}{\partial \pi_\alpha} \frac{\partial^L G}{\partial \omega^\alpha} \right] , \quad (3.7)$$

where the superscript  $L$  represents left derivation<sup>6</sup>. In what follows we will consider the derivative from the left only and for that reason we omit the superscript  $L$  on the sign of the partial derivative.

Now if we introduce the notation  $Z^M = (q^i, \omega^\alpha)$  and  $P_M = (p_i, \pi_\alpha)$  then (3.7) can be written in a more symmetric form

$$\{F, G\} = (-1)^{|F||M|} \left[ \frac{\partial^L F}{\partial Z^M} \frac{\partial^L G}{\partial P_M} - (-1)^{|M|} \frac{\partial^L F}{\partial P_M} \frac{\partial^L G}{\partial Z^M} \right] . \quad (3.8)$$

Note that these graded Poisson brackets obey the relation

$$\{F, G\} = -(-1)^{|F||G|} \{G, F\} \quad (3.9)$$

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<sup>5</sup>Since we work on the extended phase space that contains ghosts and their conjugate momenta we implicitly presume that they are included in the set of bosonic  $q^i$  or fermionic  $\omega^\alpha$  variables given above. For example, since pure spinor ghosts and their conjugate momenta are bosonic they are included in the set of variables  $q^i$ .

<sup>6</sup>The relation between left and right derivative can be found as follows. Let  $F$  is a function of Grassmann parity  $|F|$  defined on superspace labeled with  $Z^M$ . Since  $dF(Z) = dZ^M \partial_M^L F = \partial_M^R F dZ^M$  we obtain that left and right derivatives of  $F$  are related as  $(-1)^{|M||M+F|} \partial_M^L F = \partial_M^R F$ .



and

$$\{F, GH\} = \{F, G\} H + (-1)^{|H||G|} \{F, H\} G . \quad (3.10)$$

Finally we introduce the momenta conjugate to  $\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}}$  as

$$\pi_{\mu\alpha} = \frac{\delta S}{\delta \partial_0 \lambda^\alpha} = -w_{\mu\alpha} P^{\mu 0}, \quad \hat{\pi}_{\mu\hat{\alpha}} = \frac{\delta S}{\delta \partial_0 \hat{\lambda}^{\hat{\alpha}}} = -\hat{w}_{\mu\hat{\alpha}} \tilde{P}^{\mu 0} \quad (3.11)$$

with the canonical commutation relations

$$\{\lambda^\alpha(x), \pi_\beta(y)\} = \delta^\alpha_\beta \delta(x-y), \quad \{\hat{\lambda}^{\hat{\alpha}}(x), \hat{\pi}_{\hat{\beta}}(y)\} = \delta^{\hat{\alpha}}_{\hat{\beta}} \delta(x-y) . \quad (3.12)$$

Note that  $\pi_\alpha, \hat{\pi}_{\hat{\alpha}}$  carry the ghost number  $(-1, 0), (0, -1)$  respectively.

If we now return to (3.5) and multiply this expression with  $E_M^\alpha$  from the left and use the fact that  $E_A^M E_M^B = \delta_A^B$  we obtain

$$\begin{aligned} d_{\alpha\mu} P^{\mu 0} = E_\alpha^M \left( -P_M + \partial_1 Z^N B_{MN} - \Omega_{M\gamma}^\beta \lambda^\gamma w_{\beta\mu} P^{\mu 0} - \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}} \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}\mu} \tilde{P}^{\mu 0} \right) = \\ E_\alpha^M \left( -P_M + \partial_1 Z^N B_{MN} + \Omega_{M\gamma}^\beta \lambda^\gamma \pi_\beta + \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}} \hat{\lambda}^{\hat{\alpha}} \hat{\pi}_{\hat{\beta}} \right) . \end{aligned} \quad (3.13)$$

Then using (3.12) we easily get

$$\begin{aligned} \{d_{\alpha\mu} P^{\mu 0}(x), \lambda^\beta(y)\} &= -E_\alpha^M \Omega_{M\gamma}^\beta \lambda^\gamma(y) \delta(x-y) , \\ \{d_{\alpha\mu} P^{\mu 0}(x), \hat{\lambda}^{\hat{\beta}}(y)\} &= -E_\alpha^M \hat{\Omega}_{M\hat{\gamma}}^{\hat{\beta}} \hat{\lambda}^{\hat{\gamma}}(y) \delta(x-y) . \end{aligned} \quad (3.14)$$

These relations will be important in the next subsection.

On the other hand if we multiply (3.5) with  $E_{\hat{\alpha}}^M$  from the left and proceed in the same way as above we get

$$\hat{d}_{\hat{\alpha}\mu} \tilde{P}^{\mu 0} = E_{\hat{\alpha}}^M \left( -P_M + \partial_1 Z^N B_{MN} + \Omega_{M\gamma}^\beta \lambda^\gamma \pi_\beta + \hat{\Omega}_{M\hat{\alpha}}^{\hat{\beta}} \hat{\lambda}^{\hat{\alpha}} \hat{\pi}_{\hat{\beta}} \right) \quad (3.15)$$

and also

$$\begin{aligned} \{\hat{d}_{\hat{\alpha}\mu} \tilde{P}^{\mu 0}(x), \lambda^\beta(y)\} &= -E_{\hat{\alpha}}^M \Omega_{M\gamma}^\beta \lambda^\gamma(y) \delta(x-y) , \\ \{\hat{d}_{\hat{\alpha}\mu} \tilde{P}^{\mu 0}(x), \hat{\lambda}^{\hat{\beta}}(y)\} &= -E_{\hat{\alpha}}^M \hat{\Omega}_{M\hat{\gamma}}^{\hat{\beta}} \hat{\lambda}^{\hat{\gamma}}(y) \delta(x-y) . \end{aligned} \quad (3.16)$$

### 3.1 Equation of motion for massless fields in general background

Now we will argue that the ghost number  $(1, 1)$  function on the extended phase space that depends on  $Z^M$  only and that is in the cohomology of the BRST charges  $Q_L, Q_R$  describes on-shell states of the massless fluctuations around given general background.

At the first step we define the function-vertex of the ghost number  $(1, 1)$  in the form

$$V^{(1,1)} = \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(Z) . \quad (3.17)$$

Then we demand that this function is in the cohomology of the BRST operators (3.4), namely

$$\{Q_L, V^{(1,1)}\} = \{Q_R, V^{(1,1)}\} = 0 \quad (3.18)$$

and also that the operator (3.17) satisfies the gauge invariance

$$\delta V^{(1,1)} = \{Q_L, \Lambda^{(0,1)}\} + \{Q_R, \hat{\Lambda}^{(1,0)}\} , \quad (3.19)$$

where

$$\{Q_R, \Lambda^{(0,1)}\} = \{Q_L, \hat{\Lambda}^{(1,0)}\} = 0 , \quad (3.20)$$

and where  $\Lambda^{(0,1)}$ ,  $\hat{\Lambda}^{(1,0)}$  are superfields of the ghost numbers  $(0, 1)$ ,  $(1, 0)$  respectively so that can be written as

$$\Lambda^{(0,1)} = \hat{\lambda}^{\hat{\alpha}} \hat{\Gamma}_{\hat{\alpha}}(Z) , \hat{\Lambda}^{(1,0)} = \lambda^\alpha \Gamma_\alpha(Z) . \quad (3.21)$$

In order to calculate the Poisson bracket between  $Q_L$  and  $V$  we use (3.6) that implies

$$\{P_M(x), V_{\alpha\hat{\alpha}}(Z(y))\} = -(-1)^{|M|} \partial_M V_{\alpha\hat{\alpha}}(Z(x)) \delta(x - y) . \quad (3.22)$$

If we also use (3.14) we easily obtain

$$\begin{aligned} & \left\{ \lambda^\gamma d_{\gamma\mu}(x) P^{\mu 0}, \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(y) \right\} = \\ & \lambda^\gamma \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} E_\gamma^M \left( (-1)^{|M|} \partial_M A_{\alpha\hat{\alpha}} - \Omega_{\alpha M}^\beta A_{\beta\hat{\alpha}} - \hat{\Omega}_{\hat{\alpha} M}^{\hat{\beta}} A_{\alpha\hat{\beta}} \right) \delta(x - y) . \end{aligned} \quad (3.23)$$

Using this result and also (2.14) we get that (3.18) is equivalent to the following equation

$$\gamma_{mn pqr}^{\gamma\alpha} E_\gamma^M \left( (-1)^{|M|} \partial_M A_{\alpha\hat{\alpha}} - \Omega_{\alpha M}^\beta A_{\beta\hat{\alpha}} - \hat{\Omega}_{\hat{\alpha} M}^{\hat{\beta}} A_{\alpha\hat{\beta}} \right) = 0 . \quad (3.24)$$

In the same way we proceed with  $Q_R$  and we get

$$\gamma_{mn pqr}^{\hat{\gamma}\hat{\alpha}} E_{\hat{\gamma}}^M \left( (-1)^{|M|} \partial_M A_{\alpha\hat{\alpha}} - \Omega_{\alpha M}^\beta A_{\beta\hat{\alpha}} - \hat{\Omega}_{\hat{\alpha} M}^{\hat{\beta}} A_{\alpha\hat{\beta}} \right) = 0 . \quad (3.25)$$

We can argue, following [10] that these equations correctly describe the on-shell fluctuations around given supergravity solutions.

Let us now consider the action of the BRST current on the operator of the ghost number  $(M, N)$  that has the form

$$\Phi = \lambda^{\alpha_1} \dots \lambda^{\alpha_M} \hat{\lambda}^{\hat{\beta}_1} \dots \hat{\lambda}^{\hat{\beta}_N} A_{\alpha_1 \dots \alpha_M \hat{\beta}_1 \dots \hat{\beta}_N}(Z) . \quad (3.26)$$

Since the function (3.26) does not depend on  $P_M$  it is again easy to calculate the action of the BRST charges (3.4) on it and we get

$$\begin{aligned} \{Q_L, \Phi(y)\} &= \lambda^\kappa \lambda^{\alpha_1} \dots \lambda^{\alpha_M} \hat{\lambda}^{\hat{\beta}_1} \dots \hat{\lambda}^{\hat{\beta}_M} \nabla_\kappa A_{\alpha_1 \dots \alpha_M \hat{\beta}_1 \dots \hat{\beta}_N}(y) \equiv \tilde{\Phi}^{M+1, N}(y) , \\ \{Q_R, \Phi(y)\} &= \lambda^\kappa \lambda^{\alpha_1} \dots \lambda^{\alpha_M} \hat{\lambda}^{\hat{\kappa}} \hat{\lambda}^{\hat{\beta}_1} \dots \hat{\lambda}^{\hat{\beta}_M} \nabla_{\hat{\kappa}} A_{\alpha_1 \dots \alpha_M \hat{\beta}_1 \dots \hat{\beta}_N}(y) \equiv \tilde{\Phi}^{M, N+1}(y) , \end{aligned} \quad (3.27)$$

where  $\tilde{\Phi}^{M+1, N}(Z)$  is a function of the ghost number  $M+1, N$  while  $\Phi^{M, N+1}(Z)$  is the function of the ghost number  $M, N+1$ . In the previous expression the action of  $\nabla_\kappa$  on  $A_{\alpha_1 \dots \alpha_M \hat{\beta}_1 \dots \hat{\beta}_N}$  has the form

$$\begin{aligned} \nabla_\kappa A_{\alpha_1 \dots \alpha_M \hat{\beta}_1 \dots \hat{\beta}_N} &= E_\kappa^M ((-1)^{|M|} \partial_M A_{\alpha_1 \dots \alpha_M \hat{\beta}_1 \dots \hat{\beta}_N} - \\ &\Omega_{M\alpha_1}^\gamma A_{\gamma\alpha_2 \dots \alpha_M \hat{\beta}_1 \dots \hat{\beta}_N} \dots - \Omega_{M\alpha_M}^\gamma A_{\alpha_1 \dots \alpha_{M-1} \gamma \hat{\beta}_1 \dots \hat{\beta}_N} - \\ &\Omega_{M\hat{\beta}_1}^{\hat{\gamma}} A_{\alpha_1 \dots \alpha_M \hat{\gamma} \hat{\beta}_2 \dots \hat{\beta}_N} \dots - \Omega_{M\hat{\beta}_N}^{\hat{\gamma}} A_{\alpha_1 \dots \alpha_M \hat{\beta}_1 \dots \hat{\beta}_{N-1} \hat{\gamma}}) . \end{aligned} \quad (3.28)$$

Using the expression (3.27) we easily obtain that the gauge transformation of  $V^{(1,1)}$  (3.19) implies

$$\delta A_{\alpha\hat{\alpha}} = \nabla_\alpha \hat{\Gamma}_{\hat{\alpha}} + \nabla_{\hat{\alpha}} \Gamma_\alpha , \quad (3.29)$$

where we have also used (3.21). Note also that (3.20) and (3.27) together with the pure spinor constraint (2.14) imply

$$\gamma_{mnpqr}^{\alpha\beta} \nabla_\alpha \Gamma_\beta = \gamma_{mnpqr}^{\hat{\alpha}\hat{\beta}} \nabla_{\hat{\alpha}} \hat{\Gamma}_{\hat{\beta}} = 0 . \quad (3.30)$$

All these relations are correct form of the equations of motion and gauge symmetries of the massless fluctuations around given background.

However it is important to stress that the pure spinor formalism is consistent in case when the BRST operators  $Q_L, Q_R$  are nilpotent:

$$\{Q_L, Q_L\} = \{Q_R, Q_R\} = 0 \quad (3.31)$$

and also when  $Q_L, Q_R$  have vanishing Poisson bracket among themselves

$$\{Q_L, Q_R\} = 0 . \quad (3.32)$$

At the same time we have to demand that the currents  $j_{L,R}^\mu$  are separately conserved. As was shown in [12] all these conditions imply a set of constraints on the background superfields. We will not review these calculations and recommend the paper [12] for more details.

#### 4. Pure spinor action in $AdS_5 \times S_5$

In this section we review the construction of the pure spinor action in  $AdS_5 \times S_5$  with emphasis on the covariant worldsheet formulation.

As in the Matsaev-Tseytlin GS action in and  $AdS_5 \times S_5$  background [9] the action in the pure spinor formalism [4, 10, 13] is constructed from the left-invariant currents

$$J_\mu^A = (g^{-1} \partial_\mu g)^A = \partial_\mu Z^M E_M^A, \quad (4.1)$$

where  $g$  takes values in  $PSU(2, 2|4)/SO(4, 1) \times SO(5)$ ,  $A = (c, [cd], c', [c'd'], \alpha, \hat{\alpha})$  range over tangent space indices of the super-Lie algebra of  $PSU(2, 2|4)$  where  $(c, [cd])$  describe the  $SO(4, 2)$  isometries of  $AdS_5$  with  $c = 0, \dots, 4$  and  $(c', [c'd'])$  describe the  $SO(6)$  isometries of  $S^5$  with  $c' = 5, \dots, 9$ . We also preserve the notation  $\alpha$  and  $\hat{\alpha}$  for the two sixteen - component Majorana-Weyl spinors. Finally,  $Z^M$  label the target superspace and hence  $M$  ranges over the curved superspace indices ( $m = 0, \dots, 9$ ,  $\mu = 1, \dots, 16$ ,  $\hat{\mu} = 1, \dots, 16$ ).

It was also argued in [4], following [29] that the only nonzero components of  $B_{\mu\nu} = E_\mu^A E_\nu^B B_{AB}$  and  $F^{\alpha\hat{\beta}}$  are

$$B_{\alpha\hat{\beta}} = -(-1)^{|\alpha||\hat{\beta}|} B_{\hat{\beta}\alpha} = B_{\hat{\beta}\alpha} = \frac{1}{2}(ng_s)^{1/4} \delta_{\alpha\hat{\beta}}, \quad F^{\alpha\hat{\beta}} = \frac{1}{(ng_s)^{1/4}} \delta^{\alpha\hat{\beta}}, \quad (4.2)$$

where  $n$  is the value of the Ramond-Ramond flux,  $g_s$  is the string coupling constant and  $\delta_{\alpha\hat{\beta}} = (\gamma^{01234})_{\alpha\hat{\beta}}$ .

Now we are ready to write the pure spinor action in  $AdS_5 \times S_5$ . Firstly, using (4.1) we can write the embedding of  $g_{\mu\nu}$  to the worldsheet of the string as

$$g_{\mu\nu} = \partial_\mu Z^M E_M^{\underline{c}} \partial_\nu Z^N E_N^{\underline{d}} \eta_{\underline{c}\underline{d}} = J_\mu^{\underline{c}} J_\nu^{\underline{d}} \eta_{\underline{c}\underline{d}}, \quad (4.3)$$

where  $\underline{c}$  signifies either  $c$  or  $c'$ . In the same way  $\underline{cd}$  signifies either  $[cd]$  or  $[c'd']$ .

In the same way the embedding of  $b_{\mu\nu}$  can be written as

$$\epsilon^{\mu\nu} B_{\mu\nu} = \epsilon^{\mu\nu} E_\mu^A E_\nu^B B_{AB} = (ng_s)^{1/4} J_\mu^\alpha J_\nu^{\hat{\beta}} \delta_{\alpha\hat{\beta}} \quad (4.4)$$

using (4.2), antisymmetry of  $\epsilon^{\mu\nu}$  and the fact that  $J^\alpha, J^{\hat{\beta}}$  are Grassmann odd. Then the matter part of the pure spinor action in  $AdS_5 \times S_5$  takes the form

$$\begin{aligned} S = & - \int d^2x \sqrt{-\eta} \left( \frac{1}{2} \eta^{\mu\nu} g_{\mu\nu} - \frac{\epsilon^{\mu\nu}}{2} b_{\mu\nu} + F^{\alpha\hat{\beta}} d_{\alpha\mu} P^{\mu\nu} \hat{d}_{\hat{\beta}\nu} + \right. \\ & \left. E_\nu^\alpha d_{\beta\mu} P^{\mu\nu} + E_\nu^{\hat{\alpha}} \hat{d}_{\hat{\beta}\mu} \tilde{P}^{\mu\nu} \right) = \\ = & - \int d^2x \sqrt{-\eta} \left( \frac{1}{2} \eta^{\mu\nu} J_\mu^{\underline{c}} J_\nu^{\underline{d}} \eta_{\underline{c}\underline{d}} - (ng_s)^{1/4} \frac{\epsilon^{\mu\nu}}{2} J_\mu^\alpha J_\nu^{\hat{\beta}} \delta_{\alpha\hat{\beta}} + \right. \\ & \left. + \frac{1}{(ng_s)^{1/4}} \delta^{\alpha\hat{\beta}} d_{\mu\alpha} P^{\mu\nu} \hat{d}_{\nu\hat{\beta}} + J_\nu^\alpha d_{\alpha\mu} P^{\mu\nu} + J_\nu^{\hat{\alpha}} \hat{d}_{\hat{\alpha}\mu} \tilde{P}^{\mu\nu} \right) \end{aligned} \quad (4.5)$$

while the ghost action takes the form [4, 10]

$$S_{ghost} = - \int d^2x \sqrt{-\eta} (w_{\mu\alpha} P^{\mu\nu} \partial_\nu \lambda^\alpha + \hat{w}_{\mu\hat{\alpha}} \tilde{P}^{\mu\nu} \partial_\nu \hat{\lambda}^{\hat{\alpha}} + \\ + N_{\underline{cd}\mu} P^{\mu\nu} J_\nu^{[\underline{cd}]} + \hat{N}_{\underline{cd}\mu} \tilde{P}^{\mu\nu} J_\nu^{[\underline{cd}]} - N_{cd\mu} P^{\mu\nu} \hat{N}_\nu^{cd} + N_{c'd'\mu} P^{\mu\nu} N_\nu^{c'd'}) . \quad (4.6)$$

Since  $d_{\alpha\mu}$ ,  $\hat{d}_{\hat{\alpha}\mu}$  appear in (4.5) as auxiliary fields they can be integrated out and we obtain

$$P^{\mu\nu} d_{\hat{\beta}\nu} = (ng_s)^{1/4} \delta_{\hat{\beta}\alpha} P^{\mu\nu} J_\nu^\alpha , \\ \tilde{P}^{\nu\mu} d_{\mu\alpha} = -(ng_s)^{1/4} \delta_{\alpha\hat{\alpha}} \tilde{P}^{\nu\mu} J_\mu^{\hat{\alpha}} . \quad (4.7)$$

Inserting these expressions back to the (4.5) we obtain the contribution

$$-(ng_s)^{1/4} J_\mu^{\hat{\alpha}} P^{\mu\nu} \delta_{\hat{\alpha}\alpha} J_\nu^\alpha \quad (4.8)$$

and consequently the actions (4.5) and (4.6) together take the form

$$S = - \int d^2x \sqrt{-\eta} \left( \frac{1}{2} \eta^{\mu\nu} J_\mu^\underline{c} J_\nu^\underline{d} \eta_{\underline{cd}} + (ng_s)^{1/4} \eta^{\mu\nu} J_\mu^\alpha J_\nu^{\hat{\beta}} \delta_{\alpha\hat{\beta}} + (ng_s)^{1/4} \frac{1}{2} \epsilon^{\mu\nu} J_\mu^\alpha J_\nu^{\hat{\beta}} \delta_{\alpha\hat{\beta}} \right. \\ \left. - \int d^2x \sqrt{-\eta} (w_{\mu\alpha} P^{\mu\nu} \partial_\nu \lambda^\alpha + \hat{w}_{\mu\hat{\alpha}} \tilde{P}^{\mu\nu} \partial_\nu \hat{\lambda}^{\hat{\alpha}} + \right. \\ \left. + N_{\underline{cd}\mu} P^{\mu\nu} J_\nu^{[\underline{cd}]} + \hat{N}_{\underline{cd}\mu} \tilde{P}^{\mu\nu} J_\nu^{[\underline{cd}]} - (N_{cd\mu} P^{\mu\nu} \hat{N}_\nu^{cd} - N_{c'd'\mu} P^{\mu\nu} N_\nu^{c'd'}) \right) . \quad (4.9)$$

If we now perform rescaling

$$J^\underline{c} \rightarrow (ng_s)^{1/4} J^\underline{c} , J^\alpha \rightarrow (ng_s)^{1/8} J^\alpha , J^{\hat{\alpha}} \rightarrow (ng_s)^{1/8} J^{\hat{\alpha}} , J^{[\underline{cd}]} \rightarrow (ng_s)^{1/4} J^{[\underline{cd}]} \quad (4.10)$$

and also

$$(\lambda, w, \hat{\lambda}, \hat{w}) \rightarrow (Ng_s)^{1/8} (\lambda, w, \hat{\lambda}, \hat{w}) , (N, \hat{N}) \rightarrow (ng_s)^{1/4} (N, \hat{N}) \quad (4.11)$$

we obtain the action in the form

$$S = -\sqrt{ng_s} \int d^2x \sqrt{-\eta} \left( \frac{1}{2} \eta^{\mu\nu} J_\mu^\underline{c} J_\nu^\underline{d} \eta_{\underline{cd}} + \eta^{\mu\nu} J_\mu^\alpha J_\nu^{\hat{\beta}} \delta_{\alpha\hat{\beta}} + \frac{\epsilon^{\mu\nu}}{2} J_\mu^\alpha J_\nu^{\hat{\beta}} \delta_{\alpha\hat{\beta}} + \right. \\ \left. + w_{\mu\alpha} P^{\mu\nu} \partial_\nu \lambda^\alpha + \hat{w}_{\mu\hat{\alpha}} \tilde{P}^{\mu\nu} \partial_\nu \hat{\lambda}^{\hat{\alpha}} + \right. \\ \left. + N_{\underline{cd}\mu} P^{\mu\nu} J_\nu^{[\underline{cd}]} + \hat{N}_{\underline{cd}\mu} \tilde{P}^{\mu\nu} J_\nu^{[\underline{cd}]} - N_{cd\mu} P^{\mu\nu} \hat{N}_\nu^{cd} + N_{c'd'\mu} P^{\mu\nu} \hat{N}_\nu^{c'd'} \right) . \quad (4.12)$$

In what follows we omit the factor  $\sqrt{ng_s}$ .

Note that the action (4.12) can be written also in the form

$$\begin{aligned}
S = - \int d^2x \sqrt{-\eta} & \left( \frac{1}{2} \eta^{\mu\nu} J_\mu^i J_\nu^j G_{ij} + \frac{\epsilon^{\mu\nu}}{2} J_\mu^i J_\nu^j B_{ij} \right. \\
& + w_{\mu\alpha} P^{\mu\nu} \partial_\nu \lambda^\alpha + \hat{w}_{\mu\hat{\alpha}} \tilde{P}^{\mu\nu} \partial_\nu \hat{\lambda}^{\hat{\alpha}} + \\
& \left. + N_{\underline{cd}\mu} P^{\mu\nu} J_\nu^{[\underline{cd}]} + \hat{N}_{\underline{cd}\mu} \tilde{P}^{\mu\nu} J_\nu^{[\underline{cd}]} - N_{cd\mu} P^{\mu\nu} \hat{N}_\nu^{cd} + N_{c'd'\mu} P^{\mu\nu} \hat{N}_\nu^{c'd'} \right),
\end{aligned} \tag{4.13}$$

where  $i = \underline{c}, \alpha, \hat{\alpha}$  and we have defined

$$G_{ij} = \begin{pmatrix} \eta_{\underline{cd}} & 0 & 0 \\ 0 & 0 & \delta_{\alpha\hat{\beta}} \\ 0 & -\delta_{\hat{\alpha}\beta} & 0 \end{pmatrix}, \quad B_{ij} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_{\alpha\hat{\beta}} \\ 0 & \delta_{\hat{\alpha}\beta} & 0 \end{pmatrix} \tag{4.14}$$

that obey

$$G_{ij} = (-1)^{|i||j|} G_{ji}, \quad B_{ij} = -(-1)^{|i||j|} B_{ji}. \tag{4.15}$$

For letter purposes it will be useful to know the inverse matrix  $G^{ij}$

$$G^{ij} = \begin{pmatrix} \eta^{\underline{cd}} & 0 & 0 \\ 0 & 0 & -\delta^{\alpha\hat{\beta}} \\ 0 & \delta^{\hat{\alpha}\beta} & 0 \end{pmatrix}, \tag{4.16}$$

where  $\eta_{\mu\nu} \eta^{\nu\kappa} = \delta_\mu^\kappa$ ,  $\delta_{\alpha\hat{\beta}} \delta^{\hat{\beta}\gamma} = \delta_\alpha^\gamma$ ,  $\delta^{\hat{\alpha}\beta} \delta_{\beta\hat{\gamma}} = \delta_{\hat{\gamma}}^{\hat{\alpha}}$ .

The form of the action (4.13) will be useful for Hamiltonian analysis of the pure spinor action.

#### 4.1 Alternative form of the action

It turns out that for the study of the equations of motion of pure spinor string it is useful to use an alternative form of the action that was introduced recently in the paper [13]. In the covariant worldsheet formalism this action takes the form

$$\begin{aligned}
S = - \int d^2x \sqrt{-\eta} \text{Str} & \left( \frac{1}{2} \eta^{\mu\nu} \left( J_\mu^{(2)} J_\nu^{(2)} + J_\mu^{(1)} J_\nu^{(3)} + J_\mu^{(3)} J_\nu^{(1)} \right) + \right. \\
& \left. + \frac{\epsilon^{\mu\nu}}{4} \left( J_\mu^{(1)} J_\nu^{(3)} - J_\mu^{(3)} J_\nu^{(1)} \right) \right) \\
& - \int d^2x \sqrt{-\eta} \text{Str} (w_\mu P^{\mu\nu} \partial_\nu \lambda + \hat{w}_\mu \tilde{P}^{\mu\nu} \partial_\nu \hat{\lambda} + \\
& N_\mu P^{\mu\nu} J_\nu^{(0)} + \hat{N}_\mu \tilde{P}^{\mu\nu} J_\nu^{(0)} - \frac{1}{2} N_\mu P^{\mu\nu} \hat{N}_\nu - \frac{1}{2} \hat{N}_\mu \tilde{P}^{\mu\nu} N_\nu),
\end{aligned} \tag{4.17}$$

where

$$J_\mu^{(0)} = (g^{-1} \partial_\mu g)^{[\underline{cd}]} T_{[\underline{cd}]}, \quad J_\mu^{(1)} = (g^{-1} \partial_\mu g)^\alpha T_\alpha,$$

$$\begin{aligned}
J_\mu^{(2)} &= (g^{-1}\partial_\mu g)^{\underline{c}} T_{\underline{c}} , J_\mu^{(3)} = (g^{-1}\partial_\mu g)^{\hat{\alpha}} T_{\hat{\alpha}} , \\
w_\mu &= w_{\alpha\mu} G^{\alpha\hat{\beta}} T_{\hat{\beta}} , \lambda = \lambda^\alpha T_\alpha , \\
N_\mu &= -\{w_\mu, \lambda\} = -w_{\mu\beta} G^{\beta\hat{\alpha}} \lambda^\alpha \{T_{\hat{\alpha}}, T_\alpha\} = \\
&= \frac{1}{2} w_{\mu\alpha} (\gamma^{cd})_\beta^\alpha \lambda^\beta T_{[cd]} - \frac{1}{2} w_{\mu\alpha} (\gamma^{c'd'})_\beta^\alpha \lambda^\beta T_{[c'd']} = \\
&= N_\mu^{cd} T_{[cd]} - N_\mu^{c'd'} T_{[c'd']} , \\
\hat{w}_\mu &= \hat{w}_{\hat{\alpha}\mu} G^{\hat{\alpha}\beta} T_\beta , \hat{\lambda} = \hat{\lambda}^{\hat{\alpha}} T_{\hat{\alpha}} , \\
\hat{N}_\mu &= -\{\hat{w}_\mu, \hat{\lambda}\} = \\
&= \hat{N}_\mu^{cd} T_{[cd]} - \hat{N}_\mu^{c'd'} T_{[c'd']}
\end{aligned} \tag{4.18}$$

using

$$\{T_\alpha, T_{\hat{\beta}}\} = \frac{1}{2} (\gamma^{cd})_\alpha^\gamma \delta_{\gamma\hat{\beta}} T_{[cd]} - \frac{1}{2} (\gamma^{c'd'})_\alpha^\gamma \delta_{\gamma\hat{\beta}} T_{[c'd']} . \tag{4.19}$$

It can be easily shown that (4.17) is equivalent to the action (4.12) with the help of the relations

$$\text{Str}(T_{[\underline{cd}]} T_{[\underline{ef}]}) = k_{[\underline{cd}][\underline{ef}]} , \text{Str}(T_{\underline{c}} T_{\underline{d}}) = \eta_{\underline{cd}} , \text{Str}(T_\alpha, T_{\hat{\beta}}) = G_{\alpha\hat{\beta}} , \text{Str}(T_{\hat{\beta}} T_\alpha) = G_{\hat{\beta}\alpha} , \tag{4.20}$$

where  $k_{[ab][cd]}$  is defined as

$$k_{[cd][ed]} = \eta_{c[e}\eta_{f]d} , k_{[c'd'] [e'd']} = -\delta_{c'[e'}\delta_{f']d'} , k_{[cd][e'f']} = k_{[c'd'] [ef]} = 0 . \tag{4.21}$$

At this place we give a short review of some properties of  $psu(2, 2|4)$  algebra. We represent an element of this superalgebra by an even supermatrix of the form

$$G = \begin{pmatrix} A & X \\ Y & B \end{pmatrix} , \tag{4.22}$$

where  $A$  and  $B$  are matrices with Grassmann even functions while  $X$  and  $Y$  are those with Grassmann odd functions, each representing a  $4 \times 4$  matrix. (An odd supermatrix has the same form with  $A$  and  $B$  consisting of Grassmann odd functions while  $X$  and  $Y$  consisting of Grassmann even functions. An example of odd supermatrix is  $\lambda = \lambda^\alpha T_\alpha$  that has Grassmann even functions on off-diagonal blocks).

An element  $G$  of the superalgebra  $psu(2, 2|4)$  is given by a  $8 \times 8$  matrix (4.22) satisfying

$$\Sigma A^\dagger + A \Sigma = 0 , B^\dagger + B = 0 , X - i \Sigma Y^\dagger = 0 , \tag{4.23}$$

where  $\Sigma = \sigma_3 \otimes I_2$  with  $\sigma_3$  standard Pauli matrix and with  $I_2$  representing the identity matrix in 2 dimensions.

The essential feature of the superalgebra  $psu(2, 2|4)$  is that it admits a  $\mathbf{Z}_4$  automorphism such that the condition  $\mathbf{Z}_4(H) = H$  determines the maximal subgroup

to be  $SO(4, 1) \times SO(5)$  that leads to the definition of the coset for the sigma model. The  $\mathbf{Z}_4$  automorphism  $\Omega$  takes an element of  $psu(2, 2|4)$  to another  $G \rightarrow \Omega(G)$  such that

$$\Omega(G) = \begin{pmatrix} JA^T J & -JY^T J \\ JX^T J & JB^T J \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.24)$$

Since  $\Omega^4 = 1$  the eigenvalues of  $\Omega$  are  $i^p$ ,  $p = 0, 1, 2, 3$ . Therefore we can decompose the superalgebra  $G$  as

$$G = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3, \quad (4.25)$$

where  $\mathcal{H}_p$  denotes the eigenspace of  $\Omega$  such that if  $H \in \mathcal{H}_p$  then

$$\Omega(H_p) = i^p H_p. \quad (4.26)$$

The automorphism  $\Omega$  also implies an important relation

$$[H_p, H_q] \in \mathcal{H}_{p+q} \pmod{4}. \quad (4.27)$$

As we have argued above  $\Omega(\mathcal{H}_0) = \mathcal{H}_0$  determines  $\mathcal{H}_0 = SO(4, 1) \times SO(5)$  and  $\mathcal{H}_0$  is spanned by generators  $T_{[\underline{cd}]}$ .  $\mathcal{H}_2$  represents the remaining bosonic elements of the algebra and it is spanned by generators  $T_{\underline{c}}$ .  $\mathcal{H}_1$  contains fermionic elements of the algebra and it is spanned with generator  $T_\alpha$  while  $\mathcal{H}_3$  contains fermionic elements of the algebra and it is spanned with the generators  $T_{\hat{\alpha}}$ . Then we can write the current  $J_\mu$  as

$$\begin{aligned} J_\mu &= J^A T_A = J_\mu^{(0)} + J_\mu^{(1)} + J_\mu^{(2)} + J_\mu^{(3)}, \\ J_\mu^{(0)} &= J^{[\underline{cd}]} T_{[\underline{cd}]}, J_\mu^{(1)} = J_\mu^\alpha T_\alpha, \\ J_\mu^{(2)} &= J_\mu^{\underline{c}} T_{\underline{c}}, J_\mu^{(3)} = J_\mu^{\hat{\alpha}} T_{\hat{\alpha}}, \end{aligned} \quad (4.28)$$

where  $A = ([\underline{cd}], \alpha, \underline{c}, \hat{\alpha})$  and where  $J_\mu^\alpha, J_\mu^{\hat{\alpha}}$  are Grassmann odd functions while  $J_\mu^{[\underline{cd}]}, J_\mu^{\underline{c}}$  are Grassmann even functions. The generators  $T_A$  satisfy the graded algebra  $psu(2, 2|4)$

$$T_A T_B - (-1)^{|A||B|} T_B T_A = f_{AB}^C T_C. \quad (4.29)$$

The Killing form  $\langle H_p, H_q \rangle$  is also  $\mathbf{Z}_4$  invariant so that

$$\langle \Omega(H_p), \Omega(H_q) \rangle = \langle H_p, H_q \rangle \quad (4.30)$$

that using (4.26) leads to

$$\langle H_p, H_q \rangle = 0, \text{ unless } p + q = 4 \pmod{4}. \quad (4.31)$$

We define  $\langle \dots \rangle$  in terms of supertrace  $\text{Str}$  where the supertrace of the supermatrix  $G$  (4.22) is defined as

$$\text{Str}(G) = \text{Tr} A - \text{Tr} B \quad (4.32)$$



if  $G$  is an even supermatrix and

$$\text{Str}(G) = \text{Tr}A + \text{Tr}B \quad (4.33)$$

if  $G$  is odd supermatrix. Note that the supertrace of two matrices  $G, H$  obey the relation

$$\text{Str}GH = (-1)^{|G||H|}\text{Str}HG, \quad (4.34)$$

where  $|X| = 1$  if  $X$  is odd matrix. This result will be important below when we will discuss the variation of the ghost action that contains Grassmann odd matrices.

The metric of the algebra is defined as

$$G_{AB} = \text{Str}(T_A T_B) \quad (4.35)$$

and the relation (4.31) implies that the only nonzero elements are  $G_{[\underline{cd}][\underline{ef}]}$ ,  $G_{\underline{cd}}$ ,  $G_{\alpha\hat{\beta}} = -G_{\hat{\beta}\alpha}$ . The structure constant of the  $psu(2, 2|4)$  algebra also possess the graded anti-symmetry property

$$f_{AB}^D G_{DC} = -(-1)^{|A||B|} f_{BA}^D G_{DC} = -(-1)^{|B||C|} f_{AC}^D G_{DB}. \quad (4.36)$$

The form of the action (4.17) is very useful since now we can easily determine the equation of motion from it. Let us consider the variation of the group element  $g$  as  $\delta g = g\delta X$ . Then the variation of the current  $J = g^{-1}dg$  is equal to

$$\delta J_\mu = -g^{-1}\delta g g^{-1}\partial_\mu g + g^{-1}\partial_\mu \delta g = \partial_\mu \delta X + [J_\mu, \delta X]. \quad (4.37)$$

In order to find the variations of the currents  $J^{(x)}$ ,  $x = 0, 1, 2, 3$  we use the relation (4.27) when we also decompose  $\delta X$  as  $\delta X = \delta X^{(0)} + \delta X^{(1)} + \delta X^{(2)} + \delta X^{(3)}$  where  $\delta X^{(i)} \in \mathcal{H}_i$ . Then we obtain

$$\begin{aligned} \delta J^{(0)} &= d\delta X^{(0)} + [J^{(0)}, \delta X^{(0)}] + [J^{(1)}, \delta X^{(3)}] + [J^{(2)}, \delta X^{(2)}] + [J^{(3)}, \delta X^{(1)}], \\ \delta J^{(1)} &= d\delta X^{(1)} + [J^{(0)}, \delta X^{(1)}] + [J^{(1)}, \delta X^{(0)}] + [J^{(2)}, \delta X^{(3)}] + [J^{(3)}, \delta X^{(2)}], \\ \delta J^{(2)} &= d\delta X^{(2)} + [J^{(2)}, \delta X^{(0)}] + [J^{(1)}, \delta X^{(1)}] + [J^{(0)}, \delta X^{(2)}] + [J^{(3)}, \delta X^{(3)}], \\ \delta J^{(3)} &= d\delta X^{(3)} + [J^{(0)}, \delta X^{(3)}] + [J^{(1)}, \delta X^{(2)}] + [J^{(2)}, \delta X^{(1)}] + [J^{(3)}, \delta X^{(0)}]. \end{aligned} \quad (4.38)$$

Using these results we can easily perform the variation of the action (4.17) and we derive the equations of motion

$$\begin{aligned} \eta^{\mu\nu} [J_\mu^{(1)}, J_\nu^{(2)}] + (\eta^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu})(\partial_\mu J_\nu^{(3)} + [J_\mu^{(0)}, J_\nu^{(3)}]) + (\eta^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu}) [J_\nu^{(2)}, J_\mu^{(1)}] + \\ + ([J_\nu^{(3)}, N_\mu] P^{\mu\nu} + [J_\nu^{(3)}, \hat{N}_\mu] \tilde{P}^{\mu\nu}) = 0, \\ \eta^{\mu\nu} [J_\mu^{(3)}, J_\nu^{(2)}] + (\eta^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu}) [J_\mu^{(3)}, J_\nu^{(2)}] + (\eta^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu})(\partial_\nu J_\mu^{(1)} + [J_\nu^{(0)}, J_\mu^{(1)}] + \end{aligned}$$

$$\begin{aligned}
& + \left( [J_\nu^{(1)}, N_\mu] P^{\mu\nu} + [J_\nu^{(1)}, \hat{N}_\mu] \tilde{P}^{\mu\nu} \right) = 0 , \\
\partial_\mu J_\nu^{(2)} \eta^{\mu\nu} + \eta^{\mu\nu} [J_\mu^{(0)}, J_\nu^{(2)}] + (\eta^{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu}) [J_\mu^{(3)}, J_\nu^{(3)}] + (\eta^{\mu\nu} + \frac{1}{2} \epsilon^{\mu\nu}) [J_\nu^{(1)}, J_\mu^{(1)}] + \\
& + \left( [J_\nu^{(2)}, N_\mu] P^{\mu\nu} + [J_\nu^{(2)}, \hat{N}_\mu] \tilde{P}^{\mu\nu} \right) = 0 ,
\end{aligned} \tag{4.39}$$

where we have taken into account the contribution from the variation of the ghost action that is equal to

$$\begin{aligned}
& \int d^2x \text{Str}(\delta X^{(1)} [J_\nu^{(3)}, N_\mu] P^{\mu\nu} + [J_\nu^{(3)}, \hat{N}_\mu] \tilde{P}^{\mu\nu}) + \\
& \delta X^{(2)} [J_\nu^{(2)}, N_\mu] P^{\mu\nu} + [J_\nu^{(2)}, \hat{N}_\mu] \tilde{P}^{\mu\nu} + \\
& \delta X^{(3)} [J_\nu^{(1)}, N_\mu] P^{\mu\nu} + [J_\nu^{(1)}, \hat{N}_\mu] \tilde{P}^{\mu\nu}) .
\end{aligned} \tag{4.40}$$

It is also important to stress that the ghost variables transform non-trivially for the gauge transformations  $\delta X^{(0)} \in \mathcal{H}_0$

$$\begin{aligned}
\delta \lambda &= - [\delta X^{(0)}, \lambda] , \delta w_\mu = - [\delta X^{(0)}, w_\mu] , \\
\delta \hat{\lambda} &= - [\delta X^{(0)}, \hat{\lambda}] , \delta \hat{w}_\mu = - [\delta X^{(0)}, \hat{w}_\mu] .
\end{aligned} \tag{4.41}$$

These relations also determine the variation of the currents  $N, \hat{N}$  as

$$\delta N_\mu = [N_\mu, \delta X^{(0)}] , \delta \hat{N}_\mu = [\hat{N}_\mu, \delta X^{(0)}] . \tag{4.42}$$

Using this fact it is easy to see that the variation of the action with respect to  $\delta X^{(0)}$  vanishes and this result confirms the invariance of the action under gauge transformations from  $SO(4, 1) \times SO(5)$ .

We can simplify the equations (4.39) using the obvious identity  $dJ + J \wedge J = 0$  that using (4.27) can be written as

$$\begin{aligned}
\partial_\mu J_\nu^{(0)} - \partial_\nu J_\mu^{(0)} + [J_\mu^{(0)}, J_\nu^{(0)}] + [J_\mu^{(1)}, J_\nu^{(3)}] + [J_\mu^{(3)}, J_\nu^{(1)}] + [J_\mu^{(2)}, J_\nu^{(2)}] &= 0 , \\
\partial_\mu J_\nu^{(1)} - \partial_\nu J_\mu^{(1)} + [J_\mu^{(0)}, J_\nu^{(1)}] + [J_\mu^{(1)}, J_\nu^{(0)}] + [J_\mu^{(3)}, J_\nu^{(2)}] + [J_\mu^{(2)}, J_\nu^{(3)}] &= 0 , \\
\partial_\mu J_\nu^{(2)} - \partial_\nu J_\mu^{(2)} + [J_\mu^{(1)}, J_\nu^{(1)}] + [J_\mu^{(3)}, J_\nu^{(3)}] + [J_\mu^{(0)}, J_\nu^{(2)}] + [J_\mu^{(2)}, J_\nu^{(0)}] &= 0 , \\
\partial_\mu J_\nu^{(3)} - \partial_\nu J_\mu^{(3)} + [J_\mu^{(0)}, J_\nu^{(3)}] + [J_\mu^{(3)}, J_\nu^{(0)}] + [J_\mu^{(1)}, J_\nu^{(2)}] + [J_\mu^{(2)}, J_\nu^{(1)}] &= 0 .
\end{aligned} \tag{4.43}$$

For example, if we consider the last equation in (4.43) in the form

$$\epsilon^{\mu\nu} D_\mu J_\nu^{(3)} = -\epsilon^{\mu\nu} [J_\mu^{(1)}, J_\nu^{(2)}] \tag{4.44}$$

and use it in the first equation in (4.39) we obtain

$$\tilde{P}^{\mu\nu} D_\mu J_\nu^{(3)} + [J_\nu^{(3)}, N_\mu] P^{\mu\nu} + [J_\nu^{(3)}, \hat{N}_\mu] \tilde{P}^{\mu\nu} = 0 . \quad (4.45)$$

In the same we proceed with the second equation in (4.39) and we get

$$P^{\mu\nu} D_\mu J_\nu^{(1)} + [J_\nu^{(1)}, N_\mu] P^{\mu\nu} + [J_\nu^{(1)}, \hat{N}_\mu] \tilde{P}^{\mu\nu} = 0 . \quad (4.46)$$

Finally, using (4.43) we can show that third equation in (4.39) implies

$$P^{\mu\nu} D_\mu J_\nu^{(2)} - \epsilon^{\mu\nu} [J_\mu^{(1)}, J_\nu^{(1)}] + [J_\nu^{(2)}, N_\mu] P^{\mu\nu} + [J_\nu^{(2)}, \hat{N}_\mu] \tilde{P}^{\mu\nu} = 0 \quad (4.47)$$

and

$$\tilde{P}^{\mu\nu} D_\mu J_\nu^{(2)} + \epsilon^{\mu\nu} [J_\mu^{(3)}, J_\nu^{(3)}] + [J_\nu^{(2)}, N_\mu] P^{\mu\nu} + [J_\nu^{(2)}, \hat{N}_\mu] \tilde{P}^{\mu\nu} = 0 . \quad (4.48)$$

In what follows we also need the equation of motions for ghost that take the form

$$\begin{aligned} P^{\mu\nu} \left( \partial_\nu \lambda + [J_\nu^{(0)}, \lambda] \right) + \frac{1}{2} P^{\mu\nu} [\lambda, \hat{N}_\nu] + \frac{1}{2} [\lambda, \hat{N}_\nu] \tilde{P}^{\nu\mu} &= 0 , \\ \tilde{P}^{\mu\nu} \left( \partial_\nu \hat{\lambda} + [J_\nu^{(0)}, \hat{\lambda}] \right) + \frac{1}{2} [\hat{\lambda}, N_\nu] P^{\nu\mu} + \frac{1}{2} \tilde{P}^{\mu\nu} [\hat{\lambda}, N_\nu] &= 0 . \end{aligned} \quad (4.49)$$

With the help of these equations of motion we are ready to study the properties of BRST currents. Firstly, let us consider current

$$j_R^\mu = \text{Str}(\hat{\lambda} P^{\mu\nu} J_\nu^{(1)}) \quad (4.50)$$

and calculate its divergence

$$\begin{aligned} \partial_\mu j_R^\mu &= \text{Str}(\tilde{P}^{\nu\mu} \partial_\mu \hat{\lambda} J_\nu^{(1)} + \hat{\lambda} P^{\mu\nu} \partial_\mu J_\nu^{(1)}) = \\ &= \text{Str}(-\tilde{P}^{\nu\mu} [J_\mu^{(0)}, \hat{\lambda}] - \tilde{P}^{\nu\mu} [\hat{\lambda}, N_\mu]) J_\nu^{(1)} + \\ &+ \text{Str} \hat{\lambda} (-P^{\mu\nu} [J_\mu^{(0)}, J_\nu^{(1)}] - \tilde{P}^{\mu\nu} [J_\mu^{(1)}, N_\nu] - P^{\mu\nu} [J_\mu^{(1)}, \hat{N}_\nu]) = \\ &= -P^{\mu\nu} \text{Str} \hat{\lambda} [J_\mu^{(1)}, \hat{N}_\nu] = P^{\mu\nu} \text{Str} J_\mu^{(1)} [\hat{\lambda}, \hat{N}_\nu] \end{aligned} \quad (4.51)$$

using  $\tilde{P}^{\mu\nu} = P^{\nu\mu}$ . Now we can write this result as

$$\begin{aligned} [\hat{\lambda}, \hat{N}_\mu] &= -[\hat{\lambda} \hat{\lambda}, \hat{w}_\mu] = -\frac{1}{2} [\hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} (T_{\hat{\alpha}} T_{\hat{\beta}} + T_{\hat{\beta}} T_{\hat{\alpha}}), \hat{w}_\mu] = \\ &= -\frac{1}{2} \hat{\lambda}^{\hat{\alpha}} \lambda^{\hat{\beta}} \gamma_{\hat{\alpha}\hat{\beta}}^\varepsilon [T_\varepsilon, \hat{w}_\mu] = -\frac{1}{2} \hat{\lambda}^{\hat{\alpha}} (\gamma^\varepsilon)_{\hat{\alpha}\hat{\beta}} \hat{\lambda}^{\hat{\beta}} [T_\varepsilon, \hat{w}_\mu] . \end{aligned} \quad (4.52)$$

Now we immediately see that for pure spinors this expression vanishes and hence we obtain that the current  $j_R^\mu$  is conserved.

In the same way we can proceed with the second current

$$j_L^\mu = \text{Str}(\lambda \tilde{P}^{\mu\nu} J_\nu^{(3)}) \quad (4.53)$$

and consider again its divergence

$$\begin{aligned} \partial_\mu j_L^\mu &= \text{Str} \left( P^{\nu\mu} \partial_\mu \lambda J_\nu^{(3)} + \lambda \tilde{P}^{\mu\nu} \partial_\mu J_\nu^{(3)} \right) = \\ &= \text{Str} \left( \left( -P^{\nu\mu} [J_\nu^{(0)}, \hat{\lambda}] - P^{\nu\mu} [\hat{\lambda}, N_\mu] \right) J_\nu^{(3)} + \right. \\ &\quad \left. + \lambda \left( -\tilde{P}^{\mu\nu} [J_\mu^{(0)}, J_\nu^{(3)}] - [J_\nu^{(3)}, N_\mu] P^{\mu\nu} - [J_\nu^{(3)}, \hat{N}_\mu] \tilde{P}^{\mu\nu} \right) \right) = \\ &= \text{Str} P^{\mu\nu} [\lambda, N_\mu] J_\nu^{(3)} . \end{aligned} \quad (4.54)$$

In the same way as above we can again show that this expression is zero for pure spinor. Hence we can define two conserved charges

$$\begin{aligned} Q_R &= \int dx j_R^0 = \int dx \text{Str} \hat{\lambda} P^{0\nu} J_\nu^{(1)} , \\ Q_L &= \int dx j_L^0 = \int dx \text{Str} \lambda \tilde{P}^{0\nu} J_\nu^{(3)} \end{aligned} \quad (4.55)$$

that are time independent as follows from the analysis performed above.

## 4.2 Hamiltonian analysis

In this section we present the first step in the Hamiltonian analysis of the pure spinor action in  $AdS_5 \times S_5$ . It turns out that for these purposes the action (4.13) is very convenient. Firstly, using the explicit form of the currents

$$J_\mu^i = \partial_\mu Z^M E_M^i , J_\mu^{cd} = \partial_\mu Z^M \Omega_M^{[cd]} \quad (4.56)$$

we obtain

$$\begin{aligned} \eta^{\mu\nu} J_\mu^i G_{ij} J_\nu^j &= \eta^{\mu\nu} \partial_\mu Z^M E_M^i G_{ij} \partial_\nu Z^N E_N^j , \\ \epsilon^{\mu\nu} J_\mu^i J_\nu^j B_{ij} &= \epsilon^{\mu\nu} \partial_\mu Z^M \partial_\nu Z^N B_{MN} , \end{aligned} \quad (4.57)$$

where

$$\begin{aligned} B_{MN} &= (-1)^{|N||M+i|} E_M^i B_{ij} E_N^j = \\ &= -(-1)^{|M||N|} B_{NM} . \end{aligned} \quad (4.58)$$

Then the action (4.13) takes the form

$$\begin{aligned}
S = - \int d^2x \sqrt{-\eta} & \left( \frac{1}{2} \eta^{\mu\nu} \partial_\mu Z^M E_M^i G_{ij} \partial_\nu Z^N E_N^j - \frac{\epsilon^{\mu\nu}}{2} \partial_\mu Z^M \partial_\nu Z^N B_{MN} \right. \\
& + w_{\mu\alpha} P^{\mu\nu} \partial_\nu \lambda^\alpha + \hat{w}_{\mu\hat{\alpha}} \tilde{P}^{\mu\nu} \partial_\nu \hat{\lambda}^{\hat{\alpha}} + \\
& \left. + N_{\underline{cd}\mu} P^{\mu\nu} \partial_M Z^M \Omega_M^{[\underline{cd}]} + \hat{N}_{\underline{cd}\mu} \tilde{P}^{\mu\nu} \partial_M Z^M \Omega_M^{[\underline{cd}]} - N_{cd\mu} P^{\mu\nu} \hat{N}_\nu^{cd} + N_{c'd'\mu} P^{\mu\nu} N_\nu^{c'd'} \right).
\end{aligned} \tag{4.59}$$

Then we easily obtain the momenta  $P_M, \pi_\alpha, \hat{\pi}_{\hat{\alpha}}$  as

$$\begin{aligned}
P_M &= \frac{\delta^L S}{\delta^L \partial_0 Z^M} = E_M^i G_{ij} \partial_0 Z^N E_N^j + \partial_1 Z^N B_{MN} - \\
&\quad - N_{\underline{cd}\mu} P^{\mu 0} \Omega_M^{[\underline{cd}]} - \hat{N}_{\underline{cd}\mu} \tilde{P}^{\mu 0} \Omega_M^{[\underline{cd}]} , \\
\pi_\alpha &= \frac{\delta S}{\delta \partial_0 \lambda^\alpha} = -w_{\alpha\mu} P^{\mu 0}, \quad \hat{\pi}_{\hat{\alpha}} = \frac{\delta S}{\delta \partial_0 \hat{\lambda}^{\hat{\alpha}}} = -\hat{w}_{\hat{\alpha}\mu} \tilde{P}^{\mu 0}.
\end{aligned} \tag{4.60}$$

Our goal is to express  $J_0^i$  using the canonical variables  $P_M, Z^N$  and  $\lambda, \hat{\lambda}, \pi, \hat{\pi}$ . In fact, it is easy to see that (4.60) implies

$$\begin{aligned}
J_0^i &= G^{ij} E_j^M (P_M - \partial_1 Z^N B_{NM} - \\
&\quad - \frac{1}{2} \pi_\alpha (\gamma_{\underline{cd}})^\alpha_\beta \lambda^\beta \Omega_M^{[\underline{cd}]} - \frac{1}{2} \hat{\pi}_{\hat{\alpha}} (\gamma_{\underline{cd}})^{\hat{\alpha}}_{\hat{\beta}} \hat{\lambda}^{\hat{\beta}} \Omega_M^{[\underline{cd}]} ) \equiv G^{ij} E_j^M \mathbf{P}_M.
\end{aligned} \tag{4.61}$$

Now we are ready to study the action BRST  $Q_{R,L}$  (4.55) on the ghost number (1,1) function  $F$

$$F(Z, \lambda, \hat{\lambda}) = \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} V_{\alpha\hat{\alpha}}(Z) \tag{4.62}$$

that depends on  $Z$  only and as we will show it is related to the massless state of the string in  $AdS_5 \times S_5$  background. As the first step we use (4.61) to express the operator (4.55) as

$$\begin{aligned}
Q_R &= \int dx \hat{\lambda}^{\hat{\alpha}} G_{\hat{\alpha}\beta} P^{0\mu} J_\mu^\beta = - \int dx \hat{\lambda}^{\hat{\alpha}} G_{\hat{\alpha}\beta} [G^{\beta\hat{\alpha}} E_{\hat{\alpha}}^M \hat{P}_M + J_1^\beta], \\
Q_L &= \int dx \lambda^\alpha G_{\alpha\hat{\beta}} \tilde{P}^{0\mu} J_\mu^{\hat{\beta}} = - \int dx \lambda^\alpha G_{\alpha\hat{\beta}} [G^{\hat{\beta}\alpha} E_\alpha^M \hat{P}_M - J_1^{\hat{\beta}}].
\end{aligned} \tag{4.63}$$

Since the function  $F$  given in (4.62) does not depend on  $P_M$  the Poisson bracket of  $F$  with  $J_1^\alpha, J_1^{\hat{\alpha}}$  vanishes. Then using the Poisson brackets

$$\begin{aligned}
\{\mathbf{P}_M(x), Z^N(y)\} &= -(-1)^{|M|} \delta_M^N \delta(x-y), \\
\{\mathbf{P}_M(x), \lambda^\alpha(y)\} &= \frac{1}{2} (\gamma_{\underline{cd}})^\alpha_\beta \lambda^\beta \Omega_M^{[\underline{cd}]} \delta(x-y), \\
\{\mathbf{P}_M(x), \hat{\lambda}^{\hat{\alpha}}(y)\} &= \frac{1}{2} (\gamma_{\underline{cd}})^{\hat{\alpha}}_{\hat{\beta}} \hat{\lambda}^{\hat{\beta}} \Omega_M^{[\underline{cd}]} \delta(x-y)
\end{aligned} \tag{4.64}$$

we can easily calculate the Poisson bracket between  $Q_{R,L}$  and  $F$

$$\begin{aligned}\{Q_L, F(y)\} &= \lambda^\gamma E_\gamma^M [\lambda^\alpha \lambda^{\hat{\alpha}} (-1)^{|M|} \partial_M V_{\alpha\hat{\alpha}} - \\ &\quad - \frac{1}{2} \lambda^{\hat{\alpha}} (\gamma_{\underline{cd}})_\beta^\alpha \lambda^\beta \Omega_M^{[\underline{cd}]} V_{\alpha\hat{\alpha}} - \frac{1}{2} (\gamma_{\underline{cd}})_{\hat{\beta}}^{\hat{\alpha}} \lambda^{\hat{\beta}} \Omega_M^{[\underline{cd}]} V_{\alpha\hat{\alpha}}] = \\ \lambda^\gamma \lambda^\alpha \lambda^{\hat{\alpha}} E_\gamma^M [(-1)^{|M|} \partial_M V_{\alpha\hat{\alpha}} - \frac{1}{2} (\gamma_{\underline{cd}})_\alpha^\beta \Omega_M^{[\underline{cd}]} V_{\beta\hat{\alpha}} - \frac{1}{2} (\gamma_{\underline{cd}})_{\hat{\alpha}}^{\hat{\beta}} \Omega_M^{[\underline{cd}]} V_{\alpha\hat{\beta}}] &= \\ &= \lambda^\gamma \lambda^\alpha \lambda^{\hat{\alpha}} \nabla_\gamma V_{\alpha\hat{\alpha}} .\end{aligned}\quad (4.65)$$

In the same way we can calculate the action of  $Q_R$  on  $V$  and we get the result

$$\begin{aligned}\{Q_R, F(y)\} &= \hat{\lambda}^{\hat{\gamma}} \lambda^\alpha \lambda^{\hat{\alpha}} E_{\hat{\gamma}}^M [(-1)^{|M|} \partial_M V_{\alpha\hat{\alpha}} - \frac{1}{2} (\gamma_{\underline{cd}})_\alpha^\beta \Omega_M^{[\underline{cd}]} V_{\beta\hat{\alpha}} - \frac{1}{2} (\gamma_{\underline{cd}})_{\hat{\alpha}}^{\hat{\beta}} \Omega_M^{[\underline{cd}]} V_{\alpha\hat{\beta}}] = \\ &= \hat{\lambda}^{\hat{\gamma}} \lambda^\alpha \lambda^{\hat{\alpha}} \nabla_{\hat{\gamma}} V_{\alpha\hat{\alpha}}\end{aligned}\quad (4.66)$$

Now using the relations (2.14) we obtain from the requirements that  $F$  is in the cohomology of  $Q_R, Q_L$

$$\{Q_L, F\} = \{Q_R, F\} = 0 \quad (4.67)$$

the equations

$$\gamma_{mnpqr}^{\gamma\alpha} \nabla_\gamma V_{\alpha\hat{\alpha}} = \gamma_{mnpqr}^{\hat{\gamma}\hat{\alpha}} \nabla_{\hat{\gamma}} V_{\alpha\hat{\alpha}} = 0 . \quad (4.68)$$

As was shown in [13] these equations correctly describe the on-shell fluctuations around the  $AdS_5 \times S_5$  background. It is one of the main results of this paper that we were able to determine the action of the BRST operators  $Q_{L,R}$  on the function  $F$  from the first principles of the classical canonical formalism.

Generally, if we consider function of the ghost number  $(n, m)$  in the form

$$H^{(n,m)}(\lambda, \hat{\lambda}, Z) = \lambda^{\alpha_1} \dots \lambda^{\alpha_m} \hat{\lambda}^{\hat{\beta}_1} \dots \hat{\lambda}^{\hat{\alpha}_n} A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n}(Z) \quad (4.69)$$

we easily obtain that the action of  $Q_L, Q_R$  on it takes the form

$$\begin{aligned}\{Q_L, H^{(n,m)}\} &= \lambda^\kappa \lambda^{\alpha_1} \dots \lambda^{\alpha_m} \hat{\lambda}^{\hat{\beta}_1} \dots \hat{\lambda}^{\hat{\alpha}_n} \nabla_\kappa A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n} , \\ \{Q_R, H^{(n,m)}\} &= \hat{\lambda}^{\hat{\kappa}} \lambda^{\alpha_1} \dots \lambda^{\alpha_m} \hat{\lambda}^{\hat{\beta}_1} \dots \hat{\lambda}^{\hat{\alpha}_n} \nabla_{\hat{\kappa}} A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n} ,\end{aligned}\quad (4.70)$$

where

$$\begin{aligned}\nabla_\kappa A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n} &= E_\kappa^M ((-1)^{|M|} \partial_M A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n} - \Omega^{[\underline{cd}]}_\kappa \nabla_{[\underline{cd}]} A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n}) , \\ \nabla_{\hat{\kappa}} A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n} &= E_{\hat{\kappa}}^M ((-1)^{|M|} \partial_M A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n} - \Omega^{[\underline{cd}]}_{\hat{\kappa}} \nabla_{[\underline{cd}]} A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n}) , \\ \nabla_{[\underline{cd}]} A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n} &= \frac{1}{2} (\gamma_{cd})_{\alpha_1}^\gamma A_{\gamma \alpha_2 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_n} + \dots + (\gamma_{cd})_{\alpha_m}^\gamma A_{\alpha_1 \dots \alpha_{m-1} \gamma \hat{\beta}_1 \dots \hat{\beta}_n} + \\ &\quad \frac{1}{2} (\gamma_{cd})_{\hat{\beta}_1}^{\hat{\gamma}} A_{\alpha_1 \dots \alpha_m \hat{\gamma} \hat{\beta}_2 \dots \hat{\beta}_n} + \dots + \frac{1}{2} (\gamma_{cd})_{\hat{\beta}_n}^{\hat{\gamma}} A_{\alpha_1 \dots \alpha_m \hat{\beta}_1 \dots \hat{\beta}_{n-1} \hat{\gamma}} .\end{aligned}\quad (4.71)$$

Now we will study the consequence of the gauge invariance of the function  $F$

$$\begin{aligned}\delta F &= \{Q_L, \Lambda\} + \{Q_R, \hat{\Lambda}\} , \\ \{Q_R, \Lambda\} &= \{Q_L, \hat{\Lambda}\} = 0 ,\end{aligned}\tag{4.72}$$

where

$$\Lambda(\hat{\lambda}, Z) = \hat{\lambda}^{\hat{\alpha}} \Gamma_{\hat{\alpha}}(Z) , \hat{\Lambda}(\lambda, Z) = \lambda^{\alpha} \hat{\Gamma}_{\alpha}(Z) .\tag{4.73}$$

Using (4.70) we easily get

$$\{Q_L, \Lambda\} = \lambda^{\alpha} \hat{\lambda}^{\hat{\beta}} \nabla_{\alpha} \Gamma_{\hat{\beta}} , \{Q_R, \hat{\Lambda}\} = \lambda^{\alpha} \hat{\lambda}^{\hat{\beta}} \nabla_{\hat{\beta}} \Gamma_{\alpha}\tag{4.74}$$

and if we define  $\delta F = \lambda^{\alpha} \hat{\lambda}^{\hat{\beta}} \delta V_{\alpha\hat{\beta}}$  we obtain from (4.72) and (4.74) following transformations rule of  $V_{\alpha\hat{\beta}}$

$$\delta V_{\alpha\hat{\beta}} = \nabla_{\alpha} \Gamma_{\hat{\beta}} + \nabla_{\hat{\beta}} \hat{\Gamma}_{\alpha} .\tag{4.75}$$

Finally, using (4.70) and (2.14) the expressions on the second line in (4.72) imply

$$\gamma_{mnpqr}^{\alpha\beta} \nabla_{\alpha} \Gamma_{\beta} = 0 , \gamma_{mnpqr}^{\hat{\alpha}\hat{\beta}} \nabla_{\hat{\alpha}} \hat{\Gamma}_{\hat{\beta}} = 0 .\tag{4.76}$$

In summary, we have derived using the canonical Hamiltonian formalism that the ghost number (1,1) function  $F(\lambda, \hat{\lambda}, Z)$  defined on extended phase space describes the massless fluctuations on the  $AdS_5 \times S_5$  background. We hope that this result can serve as an additional support for the analysis performed in paper in [13].

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